Systematic adiabatic analysis of a nonlinear oscillator with inertia driven by colored noise

Michelle M. Wu
Physics Department, Princeton University, Princeton, New Jersey 08544

K. Y. R. Billah* and Masanobu Shinozuka
Program in Mechanics, Materials and Structures, Department of Civil Engineering and Operations Research, Princeton University, Princeton, New Jersey 08544

(Received 31 May 1995)

The Duffing oscillator with inertia excited by colored noise is studied and a colored Fokker-Planck equation and reduced stationary probability density function is obtained by systematic adiabatic expansion. The second-order moments are presented for different values of system parameters and numerical simulation is carried out. In the region where perturbation is valid, i.e., where $\epsilon \sim 0.1$, and when correlation time τ of noise is of the same order of system time scale $1/\gamma$, the results of adiabatic expansion agree very well with that of numerical simulation.

PACS number(s): 05.40.+j, 05.70.Ln

I. INTRODUCTION

Nonlinear oscillators that are subjected to random excitation occur in many areas of science and engineering. The study of a nonlinear system excited by white Gaussian noise is quite well understood [1]. To model physical systems realistically, however, it is imperative to take into account the correlation time of the noise, i.e., use the so-called colored noise.

Over the last decade substantial work has been carried out on nonlinear systems where the excitation is due to colored noise. Much of the work, however, examined the response behavior of the system that can be represented by one relevant variable obeying an overdamped equation of motion that is excited by a color noise. This results in a system where the effect of inertia is neglected from the very beginning [2] and two coupled first-order equations are solved: one for displacement x and the other for noise y. Even in this case, the coupled system is non-Markovian and approximate analysis must be used. Various methods along this line are outlined very well in Ref. [2].

In many situations, however, the assumption of overdamped dynamics neglecting inertia altogether is not appropriate. Great difficulties are encountered when the inertia of the nonlinear system and the correlation time of the excitation are accounted for in the analysis, as demonstrated in Refs. [2–4]. Nevertheless, the work represented by these references have contributed greatly to the understanding of the response of the nonlinear dynamic systems subjected to colored noise. This work is an attempt to obtain the effects of inertia within a limited range.

In this context, the nonlinear system model that has received great attention in the last few decades is the Duffing oscillator. It is one of the simplest nonlinear systems which nonetheless demonstrates a highly complex behavior. In stochastic studies, the Duffing oscillator has been mostly driven by white noise. However, in reality this is not always justifiable.

The problem of the Duffing oscillator driven by colored noise has not been completely solved yet and remains the source of interest among the researchers. In particular, recently, the van Kampen expansion method [5] and the modified stochastic linearization technique [6] were applied to the Duffing oscillator with the same success in evaluating the response of this oscillator under exponentially correlated colored noise. In the present work, the adiabatic expansion [7], which was previously found useful in a stability study [8,9], has been utilized for the same purpose, and the result is compared with the Monte Carlo simulation.

The system that is considered is described by an equation of the form

$$\frac{1}{\gamma}\ddot{x}+\dot{x}+dx+bx^3=y(t)\ . \tag{1}$$

A convenient model for the noise is a stationary Ornstein-Uhlenbeck process, described by the equation

$$\dot{y}(t)=-\frac{1}{\tau}y+\frac{\sqrt{Q}}{\tau}\xi(t)\ , \tag{2}$$

where $\xi(t)$ is the Gaussian white noise with $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(0)\rangle=\delta(t)$. The correlation function of $y(t)$ thus satisfies

$$\langle y(t)y(0)\rangle=\frac{Q}{2\tau}e^{-|t|/\tau}\ . \tag{3}$$

Note that $y(t)$ becomes a Gaussian white noise as $\tau\to 0$ with Q fixed, in which case $\langle y(t)\rangle=0$ and $\langle y(t)y(0)\rangle=Q\delta(t)$.

*Permanent address: Stevens Institute of Technology, Hoboken, NJ 07030.

 ©1995 The American Physical Society

II. ADIABATIC EXPANSION AND REDUCED PROBABILITY DENSITY

Usually a dynamic system has widely different response times. The behavior on a very short time scale is often of no interest. In fact, the purpose of the present paper is to consider the long-term behavior of the system. The adiabatic elimination method is consistent with this purpose of concentrating on the long time scale of the system [1, 2, 7]. However, the procedure of eliminating the fast variable is somewhat drastic. A more systematic method of deriving the reduced Fokker-Planck equation in a perturbative manner with higher corrections in powers of a small parameter is called *adiabatic expansion* [10–12]. In the present work, Wilemski's systematic analysis is followed to deal with a nonlinear oscillator excited by colored noise.

Introducing the scaled variables:

$$x'=x/\sqrt{Q}\,,\quad y'=(1/\sqrt{\gamma Q})y,\qquad k'(x')=k(x)/\sqrt{Q}\,, \tag{4}$$

where $k(x)=dx+bx^3$ is the force due to potential $V(x)$ associated with the Duffing equation

$$V(x)=\frac{d}{2}x^2+\frac{b}{4}x^4\,. \tag{5}$$

The new parameters are also introduced: $\lambda=1/(\gamma\tau)$, $\epsilon=\sqrt{1/\gamma}$. Rewriting the model in terms of the scaled variables and new parameters, one obtains the following system of ordinary differential equations,

$$\begin{aligned}\frac{dx'}{dt}&=\frac{1}{\epsilon}u\,,\\ \frac{du}{dt}&=-\frac{1}{\epsilon}k'(x')-\frac{1}{\epsilon^2}(u-y')\,,\\ \frac{dy'}{dt}&=-\frac{1}{\epsilon^2}\lambda y'+\frac{1}{\epsilon}\lambda\xi(t)\,.\end{aligned} \tag{6}$$

This system is solved for long-term behavior with large value of γ (fast relaxation) and arbitrary λ, i.e., $\lambda=O(1)$, $\epsilon\to 0$. From now on, all primes are omitted in notation for simplicity. The Fokker-Planck equation corresponding to Eq. (6) is

$$\begin{aligned}\partial p=&\frac{1}{\epsilon^2}\left[\lambda\frac{\partial}{\partial y}y+\frac{\lambda^2}{2}\frac{\partial^2}{\partial y^2}+\frac{\partial}{\partial u}(u-y)\right]p\\ &+\frac{1}{\epsilon}\left[k(x)\frac{\partial}{\partial u}-\frac{\partial}{\partial x}u\right]p\,,\end{aligned} \tag{7}$$

where p is the probability density $p(x,y,u,t)$. The reduced probability density $p(x,t)$ is marginal density defined as

$$p(x,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p(x,y,u,t)\,du\,dy\,, \tag{8}$$

and the moments $j_{n,m}$ are defined as

$$j_{n,m}(x,t)=\int u^n y^m p(x,y,u,t)\,du\,dy\,, \tag{9}$$

and in particular,

$$j_{0,0}(x,t)=p(x,t)\,. \tag{10}$$

Integrating over the Fokker-Planck equation for the joint probability $p(x,y,u,t)$, the following equation of $p(x,t)$ is obtained:

$$\frac{\partial}{\partial t}p(x,t)=-\frac{1}{\epsilon}\frac{\partial}{\partial x}j_{1,0}(x,t)\,. \tag{11}$$

By utilizing a method developed by Wilemski [11], a closed form expression for $p(x,t)$ can be derived in the form of a perturbation expansion in the parameter ϵ. For this purpose, multiply $u^n y^m$ and integrate Eq. (7) to obtain

$$\begin{aligned}\frac{\partial}{\partial t}j_{n,m}(x,t)=&-\frac{1}{\epsilon^2}(n+\lambda m)j_{n,m}\\ &-\frac{1}{\epsilon}\left[nk(x)j_{n-1,m}+\frac{\partial}{\partial x}j_{n+1,m}\right]\\ &+\frac{1}{\epsilon^2}\left[nj_{n-1,m+1}+\frac{\lambda^2}{2}m(m-1)j_{n,m-2}\right].\end{aligned} \tag{12}$$

As $\epsilon\to 0$ Eq. (12) describes a rapidly damped time evolution of the moments $j_{n,m}$ for $n,m\neq 0$ in a time scale ϵ^2, as indicated by the diagonal term in Eq. (12). The iterative solution of $j_{n,m}(x,t)$ in the asymptotic time regime $(t\to\infty)$ is then

$$\begin{aligned}j_{n,m}(x,t)=&\sum_{l=0}^{\infty}\left[\frac{-\epsilon^2}{n+\lambda m}\frac{\partial}{\partial t}\right]^l\frac{1}{n+\lambda m}\\ &\times\left\{-\epsilon\left[nk(x)j_{n-1,m}+\frac{\partial}{\partial x}j_{n+1,m}\right]\right.\\ &\left.+\left[nj_{n-1,m+1}+\frac{\lambda^2}{2}m(m-1)j_{n,m-2}\right]\right\}.\end{aligned} \tag{13}$$

The moment $j_{1,0}$ is evaluated up to a given order in ϵ by expressing it in terms of $j_{0,0}$ and its derivatives. In order to derive expressions to the first order in ϵ from Eq. (14), $j_{2,0}$ is evaluated to the zeroth order of ϵ and $j_{0,1}$ to the first order, which requires the evaluation of $j_{1,1}$ and $j_{0,2}$ to the zeroth order. Equation (13) is then simplified to

$$\begin{aligned}j_{1,0}(x,t)=&-\epsilon k(x)p(x,t)\\ &-\frac{1}{2}\frac{\epsilon}{1+\lambda}\left[\frac{\partial}{\partial x}+\lambda\frac{\partial}{\partial x}\right]p(x,t)+O(\epsilon^3)\,.\end{aligned} \tag{14}$$

Substituting Eq. (14) into Eq. (11) gives the desired equation for the reduced probability density $p(x,t)$

$$\frac{\partial}{\partial t}p(x,t)=\frac{\partial}{\partial x}\left[k(x)+\frac{1}{2}\frac{\partial}{\partial x}\right]p(x,t)\,. \tag{15}$$

The stationary solution is then

$$p_0(x)=N\exp\left\{-\frac{d}{Q}x^2-\frac{1}{2}\frac{b}{Q}x^4\right\}, \tag{16}$$

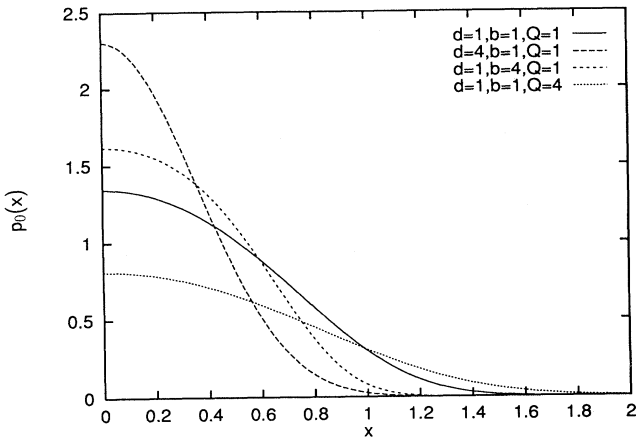

FIG. 1. Stationary probability density distribution $p_0(x)$.

which is written in terms of the unscaled and unprimed variables with N being the normalization constant.

III. MONTE CARLO SIMULATION

The numerical studies are also carried out to compare with the above analytical results. There are variety of digital simulation methods that are used in studies of statistical physics. A widely used method to solve a stochastic differential equation is the Box-Muller algorithm [13–15]. In this paper, the noise is generated from a power spectral density function [16,17], which can be considered as an alternative method. The relative merits of these two methods have been discussed in the literature [18,19]. The spectral method that is used in this work has been found to be suitable for the type of nonlinear analysis presented here [20,21,19]. The one-dimensional Gaussian noise was simulated by the following series with a large N:

$$y(t)=\sqrt{2}\sum_{n=0}^{N-1}[2S(\omega_n)\Delta\omega]^{1/2}\cos(\omega_n t+\Phi_n)\ , \qquad (17)$$

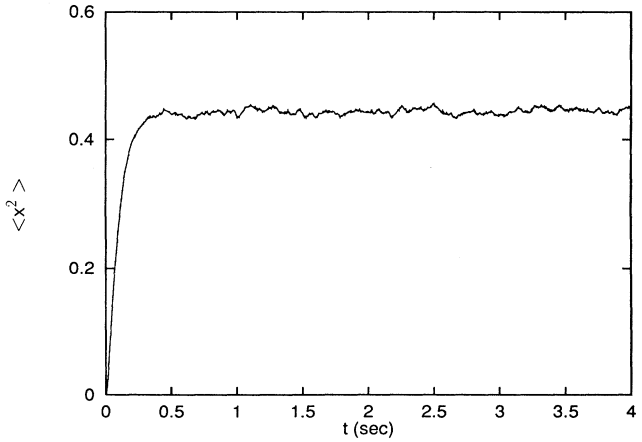

FIG. 2. Time evolution of the second-order moment $\langle x^2\rangle$ obtained from Monte Carlo simulation.

where $\omega_n=n\Delta\omega$, $n=1,2,\ldots,N$, and $\Delta\omega=\omega_u/N$. ω_u represents an upper cutoff frequency beyond which the power spectral density may be assumed to be zero. The Φ_n appearing in Eq. (17) are independent random phase angles distributed uniformly over the interval $[0,2\pi]$. The period of the stochastic process is $T_0=2\pi/\Delta\omega$.

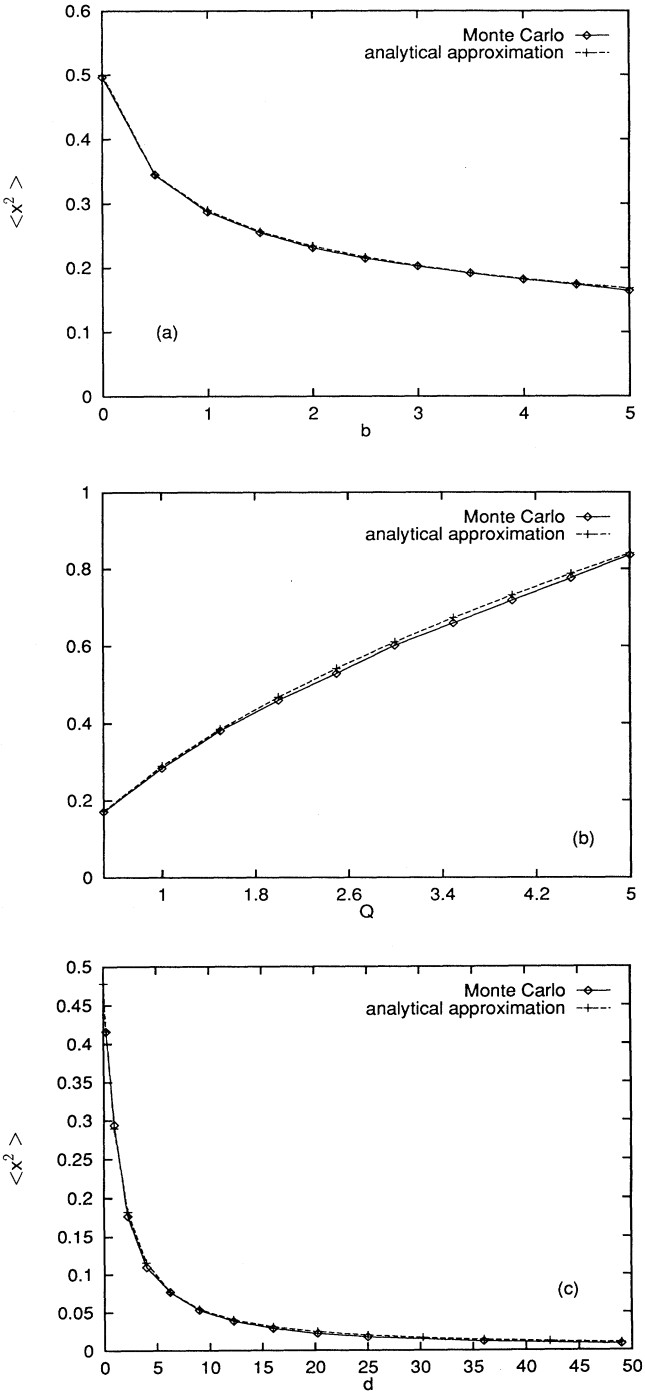

FIG. 3. Second-order moment $\langle x^2\rangle$ as functions of b, Q, and d, respectively, by theoretical analysis and Monte Carlo simulation. (a) $d=1,Q=1$; (b) $d=1,b=1$; (c) $b=1,Q=0.5$.

Corresponding to the exponential correlation function, the power spectral density $S(\omega)$ has the form

$$S(\omega)=\frac{Q}{2\pi\tau^2}\frac{1}{\omega^2+(1/\tau)^2}\ . \tag{18}$$

Using the Runge-Kutta method to integrate Eq. (1), the long-term response of the system that is excited by colored noise is found, from which the statistics, such as standard deviation or the second-order moments, are obtained.

IV. RESULTS AND DISCUSSION

To satisfy the requirement that γ be large, γ in Eq. (1) is taken as 100. At the same time, τ is chosen as 0.01 to ensure that $\lambda=1/(\tau\gamma)=O(1)$, thus $\epsilon=\sqrt{1/\gamma}=0.1$.

Figure 1 shows the stationary probability density distribution $p_0(x)$ with different values of d, b, and Q that were obtained from preceding adiabatic expansion analysis. Figure 2 shows the time evolution of the second-order moment $\langle x^2\rangle$, which was calculated by means of the Monte Carlo simulation. It can be seen that $\langle x^2\rangle$ becomes stable in a short time. Figure 3 shows that second-order moment $\langle x^2\rangle$ as functions of b, Q, and d, respectively, from both theoretical analysis and Monte Carlo simulations. The theoretical analysis means the evaluation of $\langle x^2\rangle=\int_{-\infty}^{\infty}x^2p_0(x)dx$. It can be seen clearly that the results of the theoretical analysis are in very good agreement with those of the numerical simulation.

For the limiting case in which $b=0$, the oscillator response can be solved exactly and the mean square is obtained as

$$\langle x^2\rangle=\int_{-\infty}^{+\infty}\frac{Q}{2\pi\tau^2}\frac{1}{\omega^2+(1/\tau)^2}\frac{1}{(d-\omega^2/\gamma)^2+\omega^2}d\omega\ . \tag{19}$$

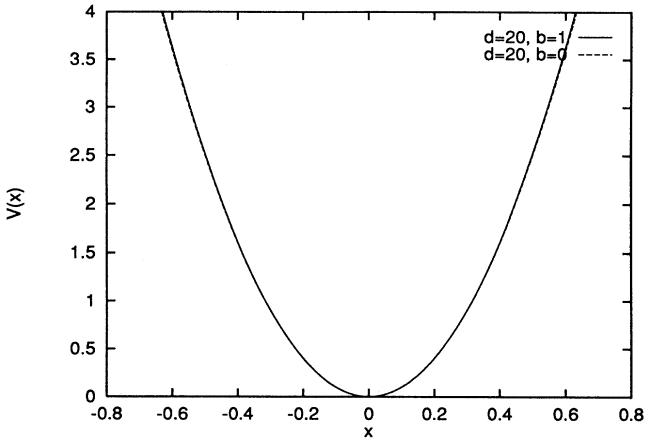

FIG. 4. Potential $V(x)=(d/2)x^2+(b/4)x^4$ with $d=20$, $b=1$ and $d=20$, $b=0$.

By using the computer algebra system MAPLE for $b=0$, $d=1$, $Q=1$, it is obtained that $\langle x^2\rangle=0.498$. The Monte Carlo simulation gives $\langle x^2\rangle=0.496$, which agrees very well with the exact value. The point at $b=0$ forms an anchor point to verify the accuracy of both the theoretical and the simulation methods. When d is large compared to b, the system can be approximated by the corresponding linear system. Figure 4 shows the potential $V(x)$ with $d=20$, $b=1$, and $d=20$, $b=0$. In the range of significance, the potentials are almost identical. The exact mean square for the linear oscillator with $d=20.25$, $Q=0.5$ is $\langle x^2\rangle=0.0112$, while the Monte Carlo simulation gives $\langle x^2\rangle=0.0111$ for $d=20.25$, $Q=0.5$, and $b=1$ (the nonlinear term is included). This agreement verifies the validity of the Monte Carlo simulation results.

[1] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences,* 2nd ed. (Springer-Verlag, Berlin Heidelberg, 1985).

[2] F. Moss and P. V. E. McClintock, *Noise in Nonlinear Dynamical Systems* (Cambridge University Press, London, 1989), Vols. 1–3.

[3] K. Lindenberg and B. J. West, *The Nonequilibrium Statistical Mechanics of Open and Closed Systems* (VCH Publisher, New York, 1990).

[4] L. H'walisz, P. Jung, P. Hänggi, P. Talkner, and L. Z. Schimansky-Geier, Phys. B **77**, 471 (1989); L. Schimansky-Geier, Phys. Lett. A **126**, 455 (1988); L. Fronzoni, P. Grigolini, P. Hänggi, F. Moss, R. Manella, and P. V. E. McClintock, Phys. Rev. A **33**, 3320 (1986); F. Marchesori, E. Menichella-Suetta, M. Pochini, and S. Santucci, Phys. Rev. A **37**, 3059 (1988).

[5] E. M. Weinstein and H. Benaroya, J. Stat. Phys. **77**, 681 (1994).

[6] G. Falsone and I. Elishakoff, Int. J. Non-Lin. Mech. **29**, 165 (1994).

[7] N. G. van Kampen, Phys. Rep. **124**, 69 (1985).

[8] R. Graham and A. Schenzle, Phys. Rev. A **26**, 1676 (1982).

[9] K. Y. R. Billah and M. Shinozuka, Phys. Rev. A **44**, R4779 (1991).

[10] R. L. Stratonovich, *Introduction to the Theory of Random Noise* (Gordon and Breach, New York, 1963).

[11] G. Wilemski, J. Stat. Phys. **14**, 153 (1976).

[12] U. M. Titulaer, Phys. **91A**, 321 (1978).

[13] R. F. Fox, Phys. Rev. A **43**, 2649 (1991).

[14] R. F. Fox, J. Stat. Phys. **54**, 1353 (1989).

[15] R. Manella and V. Palleschi, Phys. Rev. A **40**, 3381 (1989).

[16] M. Sinozuka, Comput. Struct. **2**, 85 (1972).

[17] M. Shinozuka and G. Deodatis, Appl. Mech. Rev. **44**, 191 (1991).

[18] R. Mannella and V. Palleschi, Phys. Rev. A **46**, 8028 (1992).

[19] K. Y. R. Billah and M. Shinozuka, Phys. Rev. A **46**, 8031 (1992).

[20] M. Wu, Y. K. R. Billah, and M. Shinozuka, Phys. Rev. E **51**, 995 (1995).

[21] K. Y. R. Billah and M. Shinozuka, Phys. Rev. A **42**, R7492 (1990).